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## ► To cite this version:

Nathanael Enriquez, Christophe Sabot, Marc Yor. Renewal series and square-root boundaries for Bessel processes. 2008. hal-00289151

**HAL Id: hal-00289151**

**<https://hal.science/hal-00289151>**

Preprint submitted on 19 Jun 2008

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# RENEWAL SERIES AND SQUARE-ROOT BOUNDARIES FOR BESSEL PROCESSES

NATHANAËL ENRIQUEZ, CHRISTOPHE SABOT, AND MARC YOR

**Abstract.** We show how a description of Brownian exponential functionals as a renewal series gives access to the law of the hitting time of a square-root boundary by a Bessel process. This extends classical results by Breiman and Shepp, concerning Brownian motion, and recovers by different means, extensions for Bessel processes, obtained independently by Delong and Yor.

Let  $B_t$  be the standard real valued Brownian motion and for  $\nu > 0$ , introduce the geometric Brownian motion  $\mathcal{E}_t^{(-\nu)}$  and its exponential functional  $\mathcal{A}_t^{(-\nu)}$

$$\begin{aligned}\mathcal{E}_t^{(-\nu)} &:= \exp(B_t - \nu t) \\ \mathcal{A}_t^{(-\nu)} &:= \int_0^t (\mathcal{E}_s^{(-\nu)})^2 ds.\end{aligned}$$

Lamperti's representation theorem [5] applied to  $\mathcal{E}_t^{(-\nu)}$  states

$$(0.1) \quad \mathcal{E}_t^{(-\nu)} = R_{\mathcal{A}_t^{(-\nu)}}^{(-\nu)}$$

where  $(R_u^{(-\nu)}, u \leq T_0(R^{(-\nu)}))$  denotes the Bessel process of index  $(-\nu)$  (equivalently of dimension  $\delta = 2(1 - \nu)$ ), starting at 1, which is an  $\mathbb{R}_+$ -valued diffusion with infinitesimal generator  $\mathcal{L}^{(-\nu)}$  given by

$$\mathcal{L}^{(-\nu)} f(x) = \frac{1}{2} f''(x) + \frac{1 - 2\nu}{2x} f'(x), \quad f \in C_b^2(\mathbb{R}_+^*).$$

Let us remark that, in the special case  $\nu = 1/2$ , equation (0.1) is nothing else but the Dubins-Schwarz representation of the exponential martingale  $\mathcal{E}_t^{(-1/2)}$  as Brownian motion time changed with  $\mathcal{A}_t^{(-1/2)}$ .

For a short summary of relations between Bessel processes and exponentials of Brownian motion, see e.g. Yor [10].

Let us consider now the following random variable  $Z$ , which is often called a perpetuity in the mathematical finance literature:

$$Z := \mathcal{A}_\infty^{(-\nu)} = \int_0^\infty (\mathcal{E}_s^{(-\nu)})^2 ds$$

We deduce directly from (0.1) that

$$\mathcal{A}_\infty^{(-\nu)} = T_0(R^{(-\nu)})$$

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2000 *Mathematics Subject Classification.* 60G40, 60J57.

*Key words and phrases.* Bessel processes, renewal series, exponential functionals, square-root boundaries.

where  $T_0 := \inf\{u : X_u = 0\}$ , and it is well-known (see [4], [11]), that

$$(0.2) \quad \mathcal{A}_\infty^{(-\nu)} \stackrel{(law)}{=} \frac{1}{2\gamma_\nu}$$

where  $\gamma_\nu$  is a gamma variable with parameter  $\nu$  (i.e. with density  $\frac{1}{\Gamma(\nu)}x^{\nu-1}e^{-x}\mathbf{1}_{\mathbb{R}_+}$ ).

Our main result characterizes the law of the hitting time of a parabolic boundary by  $R_u^{(-\nu)}$  which corresponds to a Bessel process of dimension  $d < 2$ .

**Theorem 1.** *Let  $0 < b < c$ , and  $\sigma := \inf\{u : (R_u^{(-\nu)})^2 = \frac{1}{c}(b + u)\}$  with  $R_0^{(-\nu)} = 1$ .*

$$(0.3) \quad E[(b + \sigma)^{-s}] = c^{-s} \frac{E[(1 + 2b\gamma_{\nu+s})^{-s}]}{E[(1 + 2c\gamma_{\nu+s})^{-s}]}, \quad \text{for any } s \geq 0$$

Proof: using the strong Markov property and the stationarity of the increments of Brownian motion, we obtain that for any stopping time  $\tau$  of the Brownian motion

$$\mathcal{A}_\infty^{(-\nu)} =: Z = \mathcal{A}_\tau^{(-\nu)} + (\mathcal{E}_\tau^{(-\nu)})^2 Z'$$

where  $Z'$  is independent of  $(\mathcal{A}_\tau^{(-\nu)}, \mathcal{E}_\tau^{(-\nu)})$  and  $Z \stackrel{(law)}{=} Z'$ .

This implies, by (0.1), that  $Z$  satisfies the following affine equation (see [8] for a survey about these equations)

$$(0.4) \quad \mathcal{A}_\infty^{(-\nu)} =: Z = \mathcal{A}_\tau^{(-\nu)} + (R_{\mathcal{A}_\tau^{(-\nu)}}^{(-\nu)})^2 Z'$$

where  $Z'$  is independent of  $(\mathcal{A}_\tau^{(-\nu)}, R_{\mathcal{A}_\tau^{(-\nu)}}^{(-\nu)})$  and  $Z \stackrel{(law)}{=} Z'$ .

Obviously,  $\sigma < T_0(R^{(-\nu)})$ . Taking now :

$$\tau = \inf\{t : (R_{\mathcal{A}_t^{(-\nu)}}^{(-\nu)})^2 = \frac{1}{c}(b + \mathcal{A}_t^{(-\nu)})\}$$

we get  $\mathcal{A}_\tau^{(-\nu)} = \sigma$ , and the identity in law

$$(0.5) \quad b + Z \stackrel{(law)}{=} (b + \sigma)(1 + \frac{Z}{c})$$

where the variables  $\sigma$  and  $Z$  on the right-hand side are independent.

As a result, we obtain the Mellin-Stieltjes transform of  $\sigma$ :

$$E[(b + \sigma)^{-s}] = c^{-s} \frac{E[(b + Z)^{-s}]}{E[(c + Z)^{-s}]}$$

But, from (0.2)

$$E[(b + \sigma)^{-s}] = c^{-s} \frac{E[(2\gamma_\nu)^s \frac{1}{(1+2b\gamma_\nu)^s}]}{E[(2\gamma_\nu)^s \frac{1}{(1+2c\gamma_\nu)^s}]}$$

which gives the result.  $\square$

One can now use the duality between the laws of Bessel processes of dimension  $d$  and  $4 - d$  to get the analogous result of Theorem 1, and recover the result of Delong [2], [3], and Yor [9] which deals with the case  $d \geq 2$ .

**Theorem 2.** *Let  $0 < b < c$ , and  $\sigma := \inf\{u : (R_u^{(\nu)})^2 = \frac{1}{c}(b + u)\}$  with  $R_0^{(\nu)} = 1$ .*

$$(0.6) \quad E[(b + \sigma)^{-s}] = c^{-s} \frac{E[(1 + 2b\gamma_s)^{-s+\nu}]}{E[(1 + 2c\gamma_s)^{-s+\nu}]}, \quad \text{for any } s \geq 0.$$

Proof : it is based on the following classical relation between the laws of the Bessel processes with indices  $\nu$  and  $-\nu$ :

$$(0.7) \quad \mathcal{P}_x^{(\nu)}|_{\mathcal{F}_t} = \frac{(X_{t \wedge T_0})^{2\nu}}{x^{2\nu}} \cdot \mathcal{P}_x^{(-\nu)}|_{\mathcal{F}_t}$$

which implies that

$$E_1^{(\nu)}[(b + \sigma)^{-s}] = E_1^{(-\nu)}[X_\sigma^{2\nu}(b + \sigma)^{-s}] = \frac{1}{c^\nu} E_1^{(-\nu)}[(b + \sigma)^{-s+\nu}]$$

Theorem 1 gives the result.  $\square$

Finally, it is easily shown, thanks to the classical representations of the Whittaker functions (see Lebedev [6]), that the right-hand sides of (0.3) and (0.6) are expressed in terms of ratios of Whittaker functions.

**Acknowledgement:** We would like to thank Daniel Dufresne for useful and enjoyable discussions on the subject.

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